

Formal symplectic geometry for Leibniz algebras

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Abstract

We study a formal symplectic geometry for anticyclic Leibniz operad and its Koszul dual operad.

1 Introduction

Let \mathfrak{g} be a finite dimensional Lie algebra. In symplectic or Poisson geometry, the Lie algebra structure on \mathfrak{g} is characterized as an odd Hamilton function, θ_{Lie} , over an even symplectic plane, $\mathcal{T}^*\Pi\mathfrak{g} = \Pi(\mathfrak{g} \times \mathfrak{g}^*)$, satisfying a Maurer-Cartan equation $\{\theta_{Lie}, \theta_{Lie}\} = 0$ (Kosmann-Schwarzbach [6], see also Roytenberg [9]). Here $\{.,.\}$ is the canonical Poisson bracket defined on the symplectic plane. The Hamiltonian system $(\mathcal{T}^*\Pi\mathfrak{g}, \theta_{Lie})$ defines a classical theory which should be quantized. We consider a noncommutative version of this Hamiltonian formalism. The noncommutative Lie algebra is known as a Leibniz algebra (Loday [8]). A Leibniz algebra is a vector space equipped with a noncommutative binary bracket satisfying the Leibniz identity

$$[x_1, [x_2, x_3]] = [[x_1, x_2], x_3] + [x_2, [x_1, x_3]].$$

The main aim of this note is to construct a Hamiltonian system which characterizes the finite dimensional Leibniz algebra. Since the Leibniz algebra is noncommutative and nonassociative, the ordinary manifold, whether graded or not, is useless for our aim. So we will use the theory of formal operad-geometry introduced by Kontsevich [5]. According to Getzler-Kapranov [3], the formal operad-geometry is a part of cyclic (co)homology theory. In general, if \mathcal{P} is a cyclic binary quadratic operad, then a cyclic (co)homology group is well-defined in the category of \mathcal{P} -algebras. Kontsevich proved in the cases of $\mathcal{P} = Com, Lie, Ass$ that if \mathcal{A} is a finite dimensional \mathcal{P} -algebra, then the cyclic cohomology theory over \mathcal{A} can be interpreted as a formal symplectic geometry via the Koszul duality theory.

We consider the case of Leibniz operad. It is known that the Leibniz operad is

anti-cyclic, although not cyclic (Chapoton [1]). Hence one can construct an anti-cyclic cohomology theory in the category of Leibniz algebras (Uchino [10]). We will prove that if \mathfrak{g} is a finite dimensional Leibniz algebra, then the anti-cyclic cohomology theory over \mathfrak{g} can be interpreted as a formal symplectic geometry as with the cyclic case.

If \mathcal{P} is a cyclic (resp. anti-cyclic) binary quadratic operad, then a finite generated (co)free $\mathcal{P}^!$ -(co)algebra is a formal $\mathcal{P}^!$ -manifold. Here $\mathcal{P}^!$ is the Koszul dual of \mathcal{P} . In the category of $\mathcal{P}^!$ -manifolds (so-called $\mathcal{P}^!$ -world), a “Lie” algebra is a \mathcal{P} -algebra and a formal function over a $\mathcal{P}^!$ -manifold is a cyclic (resp. anti-cyclic) cochain in the category of \mathcal{P} -algebras. Roughly speaking, a $\mathcal{P}^!$ -manifold is a cyclic (resp. anti-cyclic) cohomology complex in the category of \mathcal{P} -algebras. The case of $\mathcal{P}^! = \mathcal{Com}$ and $\mathcal{P} = \mathcal{Lie}$ is the classical case above.

It is well-known that the Koszul dual operad of the Leibniz operad is the Zinbiel operad ([8]). The quadratic relation of \mathcal{Zinb} is

$$x_1 * (x_2 * x_3) = (x_1 * x_2) * x_3 + (x_2 * x_1) * x_3.$$

We call the Zinbiel world a Loday world. In general, a formal function in $\mathcal{P}^!$ -world is expressed as the universal invariant bilinear form defined on the free $\mathcal{P}^!$ -algebra. Hence our main problem is to give a tensor expression of the universal invariant bilinear form on the free Zinbiel algebra. The tensor expression of the bilinear form will be used to define the canonical Poisson bracket in the Loday world. We will see that the structure of a finite dimensional Leibniz algebra is a formal function μ satisfying $\{\mu, \mu\} = 0$, where $\{.,.\}$ is a canonical Poisson bracket in the Loday world.

As an application of the formal symplectic geometry, we will study a metric tensor defined on a Leibniz algebra. In terms of generalized geometry (Hitchin [2]), a Leibniz algebra is considered to be a “generalized Lie algebra”. It is known that a natural metric tensor $g(.,.)$ defined on a generalized Lie algebra (=Leibniz algebra) satisfies

$$g([x_1, x_2], x_3) + g(x_2, [x_1, x_3]) = g(x_1, x_2 \circ x_3), \quad (1)$$

where $[.,.]$ is a Leibniz bracket and $x_2 \circ x_3 := [x_2, x_3] + [x_3, x_2]$. In the classical world, a metric tensor is not function on the symplectic plane $\mathcal{T}^*\Pi\mathfrak{g}$, because $C^\infty(\mathcal{T}^*\Pi\mathfrak{g}) = \bigwedge(\mathfrak{g} \oplus \mathfrak{g}^*)$. On the other hand, in the Loday world, a symmetric 2-tensor is a super function on the formal symplectic plane. This is an advantage that the Loday world has over the classical one. We will prove that (1) is equivalent with an invariant condition, that is, $\{\mu, g\} = 0$.

2 Leibniz and Zinbiel algebras

A (left-)Leibniz algebra is a vector space \mathfrak{g} equipped with a binary bracket $[\cdot, \cdot]$ satisfying the Leibniz identity,

$$[x_1, [x_2, x_3]] = [[x_1, x_2], x_3] + [x_2, [x_1, x_3]],$$

where $x_1, x_2, x_3 \in \mathfrak{g}$. A Zinbiel algebra is a vector space equipped with a binary product satisfying

$$x_1 * (x_2 * x_3) = (x_1 * x_2 + x_2 * x_1) * x_3.$$

The operad of Zinbiel algebras is the Koszul dual of the one of Leibniz algebras. The Leibniz algebra and the Zinbiel algebra are introduced and studied deeply by Loday ([8]). Hence they are called Loday type algebras.

Let V be a vector space. The free Zinbiel algebra over V is the tensor space $\bar{T}V := \bigoplus_{n \geq 1} V^{\otimes n}$, whose Zinbiel product is given by

$$(\dots((x_1 * x_2) * x_3) * \dots) * x_n = x_1 \otimes x_2 \otimes x_3 \otimes \dots \otimes x_n.$$

For example, $x_1 * (x_2 * x_3) = (x_1 \otimes x_2 - x_2 \otimes x_1) \otimes x_3$. By the universality of the free algebra, for any Zinbiel algebra $(Z, *)$ and for any linear map $f : V \rightarrow Z$, there exists a unique Zinbiel algebra morphism $\hat{f} : \bar{T}V \rightarrow Z$ such that the following diagram is commutative

$$\begin{array}{ccc} V & \xrightarrow{\subset} & \bar{T}V \\ & \searrow f & \downarrow \hat{f} \\ & & Z. \end{array}$$

Lemma 2.1. *If $\mathbf{x} := x_1 \otimes \dots \otimes x_a$ and $\mathbf{y} := y_1 \otimes \dots \otimes y_b$, then*

$$sh(\mathbf{x}, \mathbf{y}) = \mathbf{x} * \mathbf{y} + \mathbf{y} * \mathbf{x},$$

where $sh(\mathbf{x}, \mathbf{y})$ is the shuffle product of \mathbf{x} and \mathbf{y} .

The cofree Zinbiel coalgebra over $\Pi\mathfrak{g}$ is the tensor space $\bar{T}^c\Pi\mathfrak{g} = \bar{T}\Pi\mathfrak{g}$, whose coproduct is defined by

$$\Delta(x_1, \dots, x_n) := \sum_{\substack{1 \leq i \leq n-1 \\ \sigma}} (-1)^\sigma (x_{\sigma(1)}, \dots, x_{\sigma(i)}) \otimes (x_{\sigma(i+1)}, \dots, x_{\sigma(n-1)}, x_n),$$

where σ is an $(i, n-1-i)$ -unshuffle permutation, i.e., $\sigma(1) < \dots < \sigma(i)$ and $\sigma(i+1) < \dots < \sigma(n-1)$. Let $B : \bar{T}^i\Pi\mathfrak{g} \rightarrow \Pi\mathfrak{g}$ be an i -ary linear map on $\Pi\mathfrak{g}$. The

map is identified with a coderivation on the coalgebra. The defining identity of the coderivation is as follows. If $n \geq i$,

$$B(x_1, \dots, x_n) = \sum_{j, \sigma} (-1)^\sigma (-1)^{(i+1)j} x_{\sigma(1)} \otimes \dots \otimes x_{\sigma(j)} \otimes B(x_{\sigma(j+1)}, \dots, x_{\sigma(j+i-1)}, x_{i+j}) \otimes x_{i+j+1} \otimes \dots \otimes x_n, \quad (2)$$

where σ is a $(j, i-1)$ -unshuffle permutation and the parity of B is $i+1$. The space of the coderivations becomes a Lie algebra via the commutator. Hence, the space of the multilinear maps,

$$C_{Leib}^\bullet(\mathfrak{g}) := \text{Hom}(\bar{T}^c \Pi \mathfrak{g}, \Pi \mathfrak{g}),$$

is also a Lie algebra. If $B := [.,.]$ is binary and if B is a solution of $BB = 0$, then (\mathfrak{g}, B) becomes a Leibniz algebra and $(C_{Leib}^\bullet \mathfrak{g}, B)$ is the cohomology complex of Loday-Pirashvili [7]. We call a cochain $B \in C_{Leib}^\bullet(\mathfrak{g})$ a bar-cochain.

An invariant bilinear form in the category of Leibniz algebras is an anti-symmetric 2-form, $\langle x_1, x_2 \rangle = -\langle x_2, x_1 \rangle$, satisfying

$$\langle x_1, [x_2, x_3] \rangle = -\langle [x_2, x_1], x_3 \rangle, \quad (3)$$

$$\langle x_1, [x_2, x_3] \rangle = \langle [x_1, x_3] + [x_3, x_1], x_2 \rangle. \quad (4)$$

Suppose that \mathfrak{g} is a finite dimensional Leibniz algebra. Let \mathfrak{g}^* be the dual space of \mathfrak{g} . The coadjoint representation of \mathfrak{g} by \mathfrak{g}^* is defined by

$$\langle x_1, [x_2, a] \rangle = -\langle [x_2, x_1], a \rangle, \quad (5)$$

$$\langle x_1, [a, x_2] \rangle = \langle [x_2, x_1] + [x_2, x_1], a \rangle, \quad (6)$$

where $a \in \mathfrak{g}^*$ and $\langle ., . \rangle$ is the canonical pairing of \mathfrak{g} and \mathfrak{g}^* . The double space $\mathfrak{g} \oplus \mathfrak{g}^*$ is a symplectic plane, whose symplectic structure is defined by

$$\omega(x_1 + a_1, x_2 + a_2) := \langle x_1, a_2 \rangle - \langle x_2, a_1 \rangle. \quad (7)$$

The semi-direct product algebra $\mathfrak{g} \ltimes \mathfrak{g}^*$ is a Leibniz algebra satisfying the invariant condition (3)-(4) with respect to ω .

An invariant bilinear form in the category of Zinbiel algebras is an anti-symmetric 2-form, $\langle x_1, x_2 \rangle = -\langle x_2, x_1 \rangle$, satisfying

$$\langle x_1 * x_2, x_3 \rangle = \langle x_3 * x_2, x_1 \rangle, \quad (8)$$

$$\oint \langle x_1 * x_2, x_3 \rangle = 0, \quad (9)$$

where \oint is the cyclic summation for 1, 2, 3. The defining relations of the invariant bilinear forms were introduced by Chapton [1].

3 Anticyclic Leibniz operad

3.1 anticyclic cochains

By definition, an anticyclic $n - 1$ -cochain over \mathfrak{g} is an n -ary linear function on $\bar{T}\Pi\mathfrak{g}$ such that

$$A(x_1, \dots, x_n) = \frac{1}{n} A\left([x_1, [x_2, \dots, [x_{n-1}, x_n]]]\right), \quad (10)$$

where $[\cdot, \cdot]$ is the free Lie bracket, or commutator, over $\Pi\mathfrak{g}$. For example,

$$A(x_1, x_2, x_3) = \frac{1}{3} (A(x_1, x_2, x_3) + A(x_1, x_3, x_2) - A(x_2, x_3, x_1) - A(x_3, x_2, x_1)),$$

where $|x_i| = \text{odd}$ for each $i \in \{1, 2, 3\}$. We sometimes call the anticyclic cochain an ac cochain for short. In [10] it was proved that the set of anticyclic cochains becomes a subcomplex of the cohomology complex of Leibniz algebra.

We prove that the space of anticyclic cochains over a symplectic plane becomes an even Lie algebra.

Lemma 3.1. *The free Lie algebra over $\Pi\mathfrak{g}$ is stable for the coderivations.*

Proof. Let B be an i -ary bar-cochain in $C_{Leib}^i(\mathfrak{g})$. The case of $i = 2$ was proved in [10]. Hence we suppose that when the arity of B is $i - 1$, the lemma holds. We should compute $B[x_1, \dots, x_n]$, where $[x_1, \dots, x_n]$ is the right-normalized Lie bracket $[x_1, \dots, x_n] := [x_1, [x_2, \dots, x_n]]$. When $n = i$, the lemma obviously holds. So assume that $B[x_1, \dots, x_{n-1}]$ is an element of the free Lie algebra, where $n - 1 > i$. We have

$$[x_1, \dots, x_n] = x_1 \otimes [x_2, \dots, x_n] - (-1)^{n-1} [x_2, \dots, x_n] \otimes x_1.$$

Applying B to the first term,

$$B(x_1 \otimes [x_2, \dots, x_n]) = B_{x_1}[x_2, \dots, x_n] + (-1)^{|B|} x_1 \otimes B[x_2, \dots, x_n],$$

where $B_{x_1} := B(x_1, \cdot, \cdot, \dots, \cdot)$. Since the arity of B_{x_1} is $i - 1$ and the length of $[x_2, \dots, x_n]$ is $n - 1$, by assumption of induction $B_{x_1}[x_2, \dots, x_n]$ and $B[x_2, \dots, x_n]$ are elements of the free Lie algebra. Applying B to the second term, we have

$$B([x_2, \dots, x_n] \otimes x_1) = B[x_2, \dots, x_n] \otimes x_1 + X,$$

where X is the term which has $B(\dots, x_1)$. It is easy to prove that $X = 0$. Therefore, we obtain

$$B[x_1, \dots, x_n] = B_{x_1}[x_2, \dots, x_n] + (-1)^{|B|} [x_1, B[x_2, \dots, x_n]], \quad (11)$$

which implies the desired result. \square

From (11), we can know how $B[x_1, \dots, x_n]$ is computed. For example, if $n = 4$ and the arity of B is 3 ($|B| = \text{even}$),

$$B[x_1, x_2, x_3, x_4] = [B(x_1, x_2, x_3), x_4] + [x_3, B(x_1, x_2, x_4)] - [x_2, B(x_1, [x_3, x_4])] + [x_1, B[x_2, x_3, x_4]],$$

where $|x_i| := \text{odd}$.

Let (\mathfrak{s}, ω) be a symplectic plane, where ω is a symplectic structure on \mathfrak{s} . Let A be an anticyclic $n - 1$ -cochain over \mathfrak{s} , which is an n -linear function on $\bar{T}\Pi\mathfrak{g}$. The ac cochain is identified with a bar cochain via the symplectic structure,

$$A = (-1)^{|B|} \omega(B, -). \quad (12)$$

Let A_1 be an ac i -cochain, let A_2 an ac j -cochain and let B_1, B_2 the bar-cochains corresponding to A_1, A_2 respectively. The parities of B_1 and B_2 are $i + 1$ and $j + 1$, respectively. Define $\{A_1, A_2\}$ by

$$\{A_1, A_2\} := (-1)^{i+j} \omega([B_1, B_2], -), \quad (13)$$

which is an $i + j$ -ary linear function. From (10), we have

Lemma 3.2. *If A is an anticyclic cochain, then*

$$A(x_1, \dots, x_n) = -(-1)^{n-k+1} A(x_1, \dots, [x_k, \dots, x_n], x_{k-1}).$$

where $|x_i| := \text{odd}$.

For example, when $n = 3$,

$$A(x_1, x_2, x_3) = -A([x_2, x_3], x_1) = -A(x_2, x_3, x_1) - A(x_3, x_2, x_1).$$

Proposition 3.3. *The cochain defined in (13) is again anticyclic and the bracket $\{A_1, A_2\}$ is an even Lie bracket on the space of anticyclic cochains.*

Proof. For the sake of simplicity, we suppose that the parity of variable is even. We have

$$[B_1, B_2] = \sum B_1(\dots, B_2(\dots), \dots) - (-1)^{(i+1)(j+1)} B_2(\dots, B_1(\dots), \dots).$$

Hence

$$\begin{aligned} & (-1)^{i+j} \omega([B_1, B_2](\dots), x_n) = \\ & \sum (-1)^{j+1} A_1(\dots, B_2(\dots), \dots, x_n) - (-1)^{i+j+(i+1)(j+1)} \omega(B_2(\dots, B_1(\dots), \dots), x_n). \end{aligned}$$

where we put $n := i + j$. By Lemma 3.2,

$$\begin{aligned}
\omega(B_2(\dots, B_1(\dots), \dots), x_n) &= (-1)^{j+1} A_2(\dots, B_1(\dots), x_k, \dots, x_n) \\
&= -(-1)^{j+1} A_2(\dots, [x_k, \dots, x_n], B_1(\dots)) \\
&= -\omega(B_2(\dots, [x_k, \dots, x_n]), B_1(\dots))
\end{aligned}$$

since ω is symmetric on $\Pi\mathfrak{g}$

$$\begin{aligned}
&= -(-1)^{(i+1)(j+1)} \omega(B_1(\dots), B_2(\dots, [x_k, \dots, x_n])) \\
&= -(-1)^{(i+1)(j+1)+(i+1)} A_1(\dots, B_2(\dots, [x_k, \dots, x_n])).
\end{aligned}$$

Hence we have

$$\begin{aligned}
(-1)^{i+j} \omega([B_1, B_2](\dots), x_n) &= \\
&\sum (-1)^{j+1} A_1(\dots, B_2(\dots), \dots, x_n) + (-1)^{j+1} A_1(\dots, B_2(\dots, [x_k, \dots, x_n])).
\end{aligned}$$

From (10) and (11), one can see through that

$$(-1)^{i+j} \omega([B_1, B_2](x_1, \dots, x_{n-1}), x_n) = (-1)^{j+1} \frac{1}{i} A_1 B_2[x_1, \dots, x_n]. \quad (14)$$

This implies that $\{A_1, A_2\}$ is an anticyclic cochain. \square

We notice that $\{A_1, A_2\} \sim \omega(B_1, B_2)$. By a direct computation one can show that

Proposition 3.4.

$$\{A_1, A_2\}(x_1, \dots, x_n) = (-1)^{i+1} \omega(B_1, B_2)(1^{\otimes i} \otimes T)[x_1, \dots, x_n], \quad (15)$$

where T is the transposition of tensor, $T(x_1, \dots, x_n) := (\pm)(x_n, \dots, x_1)$.

For example, when A_1 is an ac 1-cochain and A_2 is an ac 2-cochain,

$$\begin{aligned}
(1 \otimes T)[x_1, x_2, x_3] &= x_1 \otimes T[x_2, x_3] - x_2 \otimes T(x_3 \otimes x_1) - x_3 \otimes T(x_2 \otimes x_1) \\
&= -x_1 \otimes [x_2, x_3] + x_2 \otimes (x_1 \otimes x_3) + x_3 \otimes (x_1 \otimes x_2),
\end{aligned}$$

where we put $|x_i| = \text{odd}$ for each i . Hence

$$\begin{aligned}
\{A_1, A_2\}(x_1, x_2, x_3) &= \\
&\omega(B_1(x_1), B_2[x_2, x_3]) - \omega(B_1(x_2), B_2(x_1, x_3)) - \omega(B_1(x_3), B_2(x_1, x_2)),
\end{aligned}$$

3.2 Universal invariant bilinear form

Suppose that \mathfrak{g} is a finite dimensional vector space. Let (p_i) be a linear base of $\Pi\mathfrak{g}$ and let (q^i) the dual base of $\Pi\mathfrak{g}^*$. Then the anticyclic cochain defined in (10) is expressed as follows.

$$A = \frac{1}{n} A_{i_1, \dots, i_n} [q^{i_1}, \dots, q^{i_n}]_*, \quad (16)$$

where $[\dots]_*$ is the dual of the normalized bracket $[\dots]$, which is defined as follows.

$$[x^1, \dots, x^n]_* := x^1 \otimes [x^2, \dots, x^n]_* - (-1)^{n-1} x^n \otimes [x^1, \dots, x^{n-1}]_*, \quad (17)$$

where $|x_i| := \text{odd}$ for each i .

In the following we suppose that the parity of variables are even for the sake of simplicity. For any $x_1, x_2, x_3 \in \mathfrak{g}$, the dual commutator satisfies $[x_1, x_2]_* = -[x_2, x_1]_*$ and

$$\begin{aligned} [x_1, x_2, x_3]_* &= [x_3, x_2, x_1]_*, \\ \oint [x_1, x_2, x_3]_* &= 0, \end{aligned}$$

which are the same relations as (8) and (9), respectively. Denote $\mathbf{x} := x_1 \otimes \dots \otimes x_a$ and $\mathbf{y} := y_1 \otimes \dots \otimes y_b$. We put

$$\langle \mathbf{x}, \mathbf{y} \rangle := (-1)^{b+1} [\mathbf{x}, T\mathbf{y}]_*, \quad (18)$$

where $T\mathbf{y}$ is the transposition \mathbf{y} .

Theorem 3.5. *This pairing is the universal invariant bilinear form on the free Zinbiel algebra $\bar{T}\mathfrak{g}$. Namely, if Z is a Zinbiel algebra equipped with an invariant pairing $\langle \cdot, \cdot \rangle'$ satisfying (8)-(9) and if $f : \mathfrak{g} \rightarrow Z$ is a linear map, then the universal lift of f , $\hat{f} : \bar{T}\mathfrak{g} \rightarrow Z$, preserves the bilinear form.*

First of all, we should check that $\langle \mathbf{x}, \mathbf{y} \rangle$ is antisymmetric.

Lemma 3.6. *The dual commutator is triangular, i.e.,*

$$[x_1, \dots, x_{n-1}, x_n]_* = (-1)^{n+1} [x_n, x_{n-1}, \dots, x_1]_*.$$

Proof. When $n = 2$, the identity holds. By the assumption of induction,

$$\begin{aligned} [x_1, \dots, x_n]_* &= x_1 \otimes [x_2, \dots, x_n]_* - x_n \otimes [x_1, \dots, x_{n-1}]_* \\ &= (-1)^n x_1 \otimes [x_n, \dots, x_2]_* - (-1)^n x_n \otimes [x_{n-1}, \dots, x_1]_* \\ &= (-1)^{n+1} [x_n, x_{n-1}, \dots, x_1]_*. \end{aligned}$$

□

Thanks to the lemma above, we obtain

$$\langle \mathbf{x}, \mathbf{y} \rangle = (-1)^{b+1} [\mathbf{x}, T\mathbf{y}]_* = (-1)^{b+1} (-1)^{a+b+1} [\mathbf{y}, T\mathbf{x}]_* = -(-1)^{a+1} [\mathbf{y}, T\mathbf{x}]_* = -\langle \mathbf{y}, \mathbf{x} \rangle.$$

Secondly we prove that the pairing satisfies (8).

Lemma 3.7. $sh(\mathbf{x}, \mathbf{y}) = x_1 \otimes sh(\mathbf{x}_2, \mathbf{y}) + y_1 \otimes sh(\mathbf{x}, \mathbf{y}_2)$, where $\mathbf{x}_2 := x_2 \otimes \cdots \otimes x_n$ and \mathbf{y}_2 is the same.

Denote $\mathbf{z} := z_1 \otimes \cdots \otimes z_c$. From the axiom of Zinbiel algebra,

$$\mathbf{x} * \mathbf{y} = (\mathbf{x} * \mathbf{y}^{b-1} + \mathbf{y}^{b-1} * \mathbf{x}) \otimes y_b = sh(\mathbf{x}, \mathbf{y}^{b-1}) \otimes y_b,$$

where $\mathbf{y}^{b-1} := y_1 \otimes \cdots \otimes y_{b-1}$. Hence $\langle \mathbf{x} * \mathbf{y}, \mathbf{z} \rangle = \langle sh(\mathbf{x}, \mathbf{y}^{b-1}) \otimes y_b, \mathbf{z} \rangle$. We should prove

$$(-1)^{c+1} [sh(\mathbf{x}, \mathbf{y}^{b-1}), y_b, T\mathbf{z}]_* = (-1)^{a+1} [sh(\mathbf{z}, \mathbf{y}^{b-1}), y_b, T\mathbf{x}]_*. \quad (19)$$

From Lemma above,

$$\begin{aligned} (-1)^{c+1} [sh(\mathbf{x}, \mathbf{y}^{b-1}), y_b, T\mathbf{z}]_* &= \\ &= (-1)^{c+1} \left([x_1, sh(\mathbf{x}_2, \mathbf{y}^{b-1}), y_b, T\mathbf{z}]_* + [y_1, sh(\mathbf{x}, \mathbf{y}_2^{b-1}), y_b, T\mathbf{z}]_* \right), \end{aligned} \quad (20)$$

where $\mathbf{y}_2^{b-1} := y_2 \otimes \cdots \otimes y_{b-1}$. The first term of (20) is

$$\begin{aligned} (-1)^{c+1} [x_1, sh(\mathbf{x}_2, \mathbf{y}^{b-1}), y_b, T\mathbf{z}]_* &= \\ &= (-1)^{c+1} x_1 \otimes [sh(\mathbf{x}_2, \mathbf{y}^{b-1}), y_b, T\mathbf{z}]_* + (-1)^c z_1 \otimes [x_1, sh(\mathbf{x}_2, \mathbf{y}^{b-1}), y_b, T\mathbf{z}_2]_* = \end{aligned}$$

by the assumption of induction

$$= (-1)^a x_1 \otimes [sh(\mathbf{z}, \mathbf{y}^{b-1}), y_b, T\mathbf{x}_2]_* + (-1)^c z_1 \otimes [x_1, sh(\mathbf{x}_2, \mathbf{y}^{b-1}), y_b, T\mathbf{z}_2]_*. \quad (21)$$

The second term of (20) is in the same way

$$\begin{aligned} (-1)^{c+1} [y_1, sh(\mathbf{x}, \mathbf{y}_2^{b-1}), y_b, T\mathbf{z}]_* &= \\ &= (-1)^{c+1} y_1 \otimes [sh(\mathbf{x}, \mathbf{y}_2^{b-1}), y_b, T\mathbf{z}]_* + (-1)^c z_1 \otimes [y_1, sh(\mathbf{x}, \mathbf{y}_2^{b-1}), y_b, T\mathbf{z}_2]_* = \\ &= (-1)^{a+1} y_1 \otimes [sh(\mathbf{z}, \mathbf{y}_2^{b-1}), y_b, T\mathbf{x}]_* + (-1)^c z_1 \otimes [y_1, sh(\mathbf{x}, \mathbf{y}_2^{b-1}), y_b, T\mathbf{z}_2]_*. \end{aligned} \quad (22)$$

(21)+(22) is

$$\begin{aligned} &(-1)^a x_1 \otimes [sh(\mathbf{z}, \mathbf{y}^{b-1}), y_b, T\mathbf{x}_2]_* + (-1)^c z_1 \otimes [x_1, sh(\mathbf{x}_2, \mathbf{y}^{b-1}), y_b, T\mathbf{z}_2]_* + \\ &(-1)^{a+1} y_1 \otimes [sh(\mathbf{z}, \mathbf{y}_2^{b-1}), y_b, T\mathbf{x}]_* + (-1)^c z_1 \otimes [y_1, sh(\mathbf{x}, \mathbf{y}_2^{b-1}), y_b, T\mathbf{z}_2]_* = \\ &(-1)^a x_1 \otimes [sh(\mathbf{z}, \mathbf{y}^{b-1}), y_b, T\mathbf{x}_2]_* + (-1)^{a+1} y_1 \otimes [sh(\mathbf{z}, \mathbf{y}_2^{b-1}), y_b, T\mathbf{x}]_* + \\ &(-1)^c z_1 \otimes [sh(\mathbf{x}, \mathbf{y}^{b-1}), y_b, T\mathbf{z}_2]_* = \end{aligned}$$

by the assumption of induction again

$$= (-1)^a x_1 \otimes [sh(\mathbf{z}, \mathbf{y}^{b-1}), y_b, T\mathbf{x}_2]_* + (-1)^{a+1} y_1 \otimes [sh(\mathbf{z}, \mathbf{y}_2^{b-1}), y_b, T\mathbf{x}]_* + \\ (-1)^{a+1} z_1 \otimes [sh(\mathbf{z}_2, \mathbf{y}^{b-1}), y_b, T\mathbf{x}]_*,$$

which is equal to the right-hand side of (19). Therefore,

$$\langle \mathbf{x} * \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{z} * \mathbf{y}, \mathbf{x} \rangle.$$

In the same way by using induction one can show that

$$\oint \langle \mathbf{x} * \mathbf{y}, \mathbf{z} \rangle = 0. \quad (23)$$

Finally we prove that the pairing $\langle \mathbf{x}, \mathbf{y} \rangle = (-1)^{b+1} [\mathbf{x}, T\mathbf{y}]_*$ is universal. Let Z be a Zinbiel algebra equipped with an invariant bilinear form $\langle \cdot, \cdot \rangle'$ and let $f : \mathfrak{g} \rightarrow Z$ be a linear map. We should prove that the lift \hat{f} preserves the pairing. It suffices to consider the case of $\langle x_1 \otimes \cdots \otimes x_{n-1}, x_n \rangle$. By (8)-(9),

$$\begin{aligned} \langle x_1 \otimes \cdots \otimes x_{n-1}, x_n \rangle &= \langle x_n \otimes x_{n-1}, x_1 \otimes \cdots \otimes x_{n-2} \rangle \\ &= -\langle x_1 \otimes \cdots \otimes x_{n-2}, x_n \otimes x_{n-1} \rangle \\ &= \cdots \\ &= (-1)^{n-1-i} \langle x_1 \otimes \cdots \otimes x_i, x_n \otimes \cdots \otimes x_{i+1} \rangle \\ &= (-1)^{n-1-i} \langle \mathbf{x}^{i-1} \otimes x_i, T\mathbf{x}_{i+1} \rangle \\ &= -(-1)^{n-1-i} \langle T\mathbf{x}_{i+1} * \mathbf{x}^{i-1}, x_i \rangle - (-1)^{n-1-i} \langle x_i * T\mathbf{x}_{i+1}, \mathbf{x}^{i-1} \rangle \\ &= -(-1)^{n-1-i} \langle T\mathbf{x}_{i+1} * \mathbf{x}^{i-1}, x_i \rangle - (-1)^{n-1-i} \langle \mathbf{x}^{i-1} * T\mathbf{x}_{i+1}, x_i \rangle \\ &= (-1)^{n-i} \langle sh(T\mathbf{x}_{i+1}, \mathbf{x}^{i-1}), x_i \rangle. \end{aligned}$$

On the other hand, one can show that

$$[x_1, \dots, x_n]_* = \sum_{i=1}^n (-1)^{n-i} sh(T\mathbf{x}_{i+1}, \mathbf{x}^{i-1}) \otimes x_i,$$

where $sh(\emptyset, -) = sh(-, \emptyset) = id$. We put

$$(\hat{f} \otimes f)(sh(T\mathbf{x}_{i+1}, \mathbf{x}^{i-1}) \otimes x_i) := \langle \hat{f}T\mathbf{x}_{i+1} * \hat{f}\mathbf{x}^{i-1} + \hat{f}\mathbf{x}^{i-1} * \hat{f}T\mathbf{x}_{i+1}, fx_i \rangle'.$$

Then we obtain

$$\frac{1}{n} (\hat{f} \otimes f)[x_1, \dots, x_n]_* = \langle \hat{f}\mathbf{x}, fx_n \rangle'.$$

This means that $\langle \mathbf{x}, \mathbf{y} \rangle = (-1)^{b+1} [\mathbf{x}, T\mathbf{y}]_*$ is the universal invariant bilinear form.

4 Loday world

Let \mathfrak{g} be a finite dimensional vector space (not necessarily Leibniz algebra) and let $F\mathfrak{g}$ the space of anticyclic cochains over \mathfrak{g} . Here $F\mathfrak{g} = \bigoplus_{i \geq 2} F^i\mathfrak{g}$ and $F^i\mathfrak{g}$ is the space of ac $i - 1$ -cochains.

Definition 4.1. *The triple $\Pi\mathcal{M} := (\bar{T}^c\Pi\mathfrak{g}, \bar{T}\Pi\mathfrak{g}^*, F\mathfrak{g})$ is called a formal super Zinbiel manifold or super Loday manifold, where $\bar{T}^c\Pi\mathfrak{g}$ and $\bar{T}\Pi\mathfrak{g}^*$ are the cofree Zinbiel coalgebra over $\Pi\mathfrak{g}$ and the free Zinbiel algebra over the dual space, respectively.*

- By definition a function or formal function over the manifold is an anticyclic cochain in $F\mathfrak{g}$.
- A local coordinate of $\Pi\mathcal{M}$ is by definition a linear base of $\Pi\mathfrak{g}^*$. When \mathfrak{g} is an ordinary vector space, the coordinate degree is odd.
- A vector field on $\Pi\mathcal{M}$ is by definition a bar-cochain or equivalently coderivation on the cofree coalgebra $\bar{T}^c\Pi\mathfrak{g}$.

The above definition holds for any binary quadratic cyclic or anticyclic operads. We here give a general definition of formal super operad-manifold. Let \mathcal{P} be a binary quadratic cyclic (resp. anticyclic) operad, let $\mathcal{P}^!$ the Koszul dual of \mathcal{P} and let V a finite dimensional vector space. The formal super $\mathcal{P}^!$ -manifold over V is the following data:

- $\tilde{\mathcal{P}}^{!c}\Pi V$: the cofree $\mathcal{P}^!$ -coalgebra over ΠV .
- $\tilde{\mathcal{P}}^!\Pi V$: the free $\mathcal{P}^!$ -algebra over ΠV .
- $F(V, \mathcal{P})$: the space of cyclic (resp. anticyclic) cochains over V in the category of \mathcal{P} -algebras.

The case of cyclic operad was studied in [5], in particular when $\mathcal{P} = \text{Com}, \text{Ass}, \text{Lie}$.

Let $\Pi\mathcal{M}$ be the super Loday manifold over \mathfrak{g} and let $(p_i), (q^j)$ are linear bases of $\Pi\mathfrak{g}$ and $\Pi\mathfrak{g}^*$ respectively. The base (q^j) is a local coordinate of the manifold.

Definition 4.2. *The coordinate derivation of the function on $\Pi\mathcal{M}$ is defined as follows.*

$$\frac{\partial}{\partial q^i}[x^1, \dots, x^n]_* := (\pm) \sum_{j=1}^n x^{\sigma_1} \otimes \dots \otimes x^{\sigma_{n-1}} \otimes \frac{\partial x^{\sigma(n)}}{\partial q^i}.$$

Namely, after expansion, the most right-component is derived.

The derivation is a map of $F\mathfrak{g}$ to the free Zinbiel algebra $\bar{T}\Pi\mathfrak{g}$.

Consider the symplectic plane $\mathfrak{s} := \mathfrak{g} \oplus \mathfrak{g}^*$ with the symplectic structure ω defined in (7).

Definition 4.3 (cotangent bundle). $\mathcal{T}^*\Pi\mathcal{M} := (\bar{T}^c\Pi\mathfrak{s}, \bar{T}\Pi\mathfrak{s}^*, F\mathfrak{s})$.

The canonical Poisson bracket over the cotangent bundle is defined as follows.

Definition 4.4 (Poisson bracket).

$$\{A_1, A_2\} := \sum_i (-1)^{|A_1|} \left\langle \frac{\partial A_1}{\partial p_i}, \frac{\partial A_2}{\partial q^i} \right\rangle - (-1)^{|A_1|} \left\langle \frac{\partial A_1}{\partial q^i}, \frac{\partial A_2}{\partial p_i} \right\rangle,$$

where $A_1, A_2 \in F\mathfrak{s}$ and $\langle \cdot, \cdot \rangle$ is the universal invariant bilinear form introduced in Section 3.2.

$\frac{\partial A}{\partial p_i}$ and $\frac{\partial A}{\partial q^i}$ are respectively equal to $(\pm)A(\dots, q^i)$ and $(\pm)A(\dots, p^i)$. This implies that the Poisson bracket is equivalent with the graded Lie bracket in Proposition 3.4.

Definition 4.5 (Hamiltonian vector field). Let A be a function over $\mathcal{T}^*\Pi\mathcal{M}$ or anticyclic cochain over \mathfrak{s} . The coderivation B defined by (12) is called a Hamiltonian vector field of A .

Definition 4.6 (structures). A function, θ , over $\mathcal{T}^*\Pi\mathcal{M}$ is called a structure, if it is a cubic form satisfying $\{\theta, \theta\} = 0$. A Q -structure is the Hamiltonian vector field of θ .

Let $[\cdot, \cdot]$ be a binary bracket product on \mathfrak{g} , which can be extended on \mathfrak{s} via the coadjoint action (5)-(6). We put

$$\mu := C_{ij}^k [q^i, q^j, p_k]_*,$$

where $C_{ij}^k := \omega([p_i, p_j], q_k)$.

Theorem 4.7. $\{\mu, \mu\} = 0$ if and only if $[\cdot, \cdot]$ is a Leibniz bracket.

Proof. The proof is by a direct computation. We denote $x \otimes y$ by shortly xy . Then

$$\mu = C_{ij}^k q^i q^j p_k + C_{ij}^k q^i p_k q^j - C_{ij}^k p_k q^i q^j - C_{ij}^k p_k q^j q^i.$$

It suffices to compute $\langle \frac{\partial \mu}{\partial p_a}, \frac{\partial \mu}{\partial q^a} \rangle$. By the definition of the derivation,

$$\begin{aligned} \frac{\partial \mu}{\partial p_a} &= C_{ij}^a q^i q^j \\ \frac{\partial \mu}{\partial q^a} &= C_{ia}^k q^i p_k - C_{ia}^k p_k q^i - C_{aj}^k p_k q^j \\ &= C_{ka}^l q^k p_l - C_{ka}^l p_l q^k - C_{ak}^l p_l q^k \end{aligned}$$

The first pairing is $\langle C_{ij}^a q^i q^j, C_{ka}^l q^k p_l \rangle = C_{ij}^a C_{ka}^l \langle q^i q^j, q^k p_l \rangle$. By the invariant condition,

$$\begin{aligned} \langle q^i q^j, q^k p_l \rangle &= -\langle q^k p_l, q^i q^j \rangle \\ &= \langle (q^i q^j) q^k, p_l \rangle - \langle p_l (q^i q^j), q^k \rangle \\ &= \langle q^i q^j q^k, p_l \rangle + \langle q^k (q^i q^j), p_l \rangle \\ &= \langle q^i q^j q^k, p_l \rangle + \langle q^k q^i q^j, p_l \rangle - \langle q^i q^k q^j, p_l \rangle. \end{aligned}$$

Hence

$$\begin{aligned}\langle C_{ij}^a q^i q^j, C_{ka}^l q^k p_l \rangle &= C_{ij}^a C_{ka}^l \langle q^i q^j q^k, p_l \rangle + C_{ij}^a C_{ka}^l \langle q^k q^i q^j, p_l \rangle - C_{ij}^a C_{ka}^l \langle q^i q^k q^j, p_l \rangle \\ &= C_{ij}^a C_{ka}^l \langle q^i q^j q^k, p_l \rangle + C_{jk}^a C_{ia}^l \langle q^i q^j q^k, p_l \rangle - C_{ik}^a C_{ja}^l \langle q^i q^j q^k, p_l \rangle.\end{aligned}$$

In the same way,

$$\begin{aligned}\langle C_{ij}^a q^i q^j, -C_{ka}^l p_l q^k \rangle &= -C_{ij}^a C_{ka}^l \langle q^i q^j, p_l q^k \rangle \\ &= C_{ij}^a C_{ka}^l \langle p_l q^k, q^i q^j \rangle \\ &= -C_{ij}^a C_{ka}^l \langle q^i q^j q^k, p_l \rangle\end{aligned}$$

and

$$\langle C_{ij}^a q^i q^j, -C_{ak}^l p_l q^k \rangle = -C_{ij}^a C_{ak}^l \langle q^i q^j q^k, p_l \rangle.$$

We obtain

$$\begin{aligned}\left\langle \frac{\partial \mu}{\partial p_a}, \frac{\partial \mu}{\partial q^a} \right\rangle &= C_{jk}^a C_{ia}^l \langle q^i q^j q^k, p_l \rangle - C_{ik}^a C_{ja}^l \langle q^i q^j q^k, p_l \rangle - C_{ij}^a C_{ak}^l \langle q^i q^j q^k, p_l \rangle \\ &= (C_{jk}^a C_{ia}^l - C_{ik}^a C_{ja}^l - C_{ij}^a C_{ak}^l) \langle q^i q^j q^k, p_l \rangle \\ &= ([i, [j, k]] - [j, [i, k]] - [[i, j], k]) \langle q^i q^j q^k, p_l \rangle.\end{aligned}$$

Therefore, if $[\cdot, \cdot]$ is a Leibniz bracket, then $\{\mu, \mu\} = 0$. We put

$$L_{ijk}^l := C_{jk}^a C_{ia}^l - C_{ik}^a C_{ja}^l - C_{ij}^a C_{ak}^l$$

By the definition of the pairing,

$$\begin{aligned}\langle q^i q^j q^k, p_l \rangle &= [q^i, q^j, q^k, p_l]_* \\ &= q^i \otimes [q^j, q^k, p_l]_* + p_l \otimes [q^i, q^j, q^k]_* \\ &\dots \dots \\ &= q^i \otimes q^j \otimes q^k \otimes p_l + \dots\end{aligned}$$

If $\{\mu, \mu\} = 0$, then

$$L_{ijk}^l q^i \otimes q^j \otimes q^k \otimes p_l = 0,$$

which implies that $[\cdot, \cdot]$ is Leibniz. \square

The function μ is a structure which characterizes the semi-direct product Leibniz algebra $\mathfrak{g} \ltimes \mathfrak{g}^*$. More generally, when $\mathfrak{g} \oplus \mathfrak{g}^*$ is an Abelian extension of \mathfrak{g} by \mathfrak{g}^* , the structure has the following form,

$$\theta_{Leib} := C_{ij}^k [q^i, q^j, p_k]_* + \frac{1}{3} H_{ijk} [q^i, q^j, q^k]_*$$

and $\{\theta_{Leib}, \theta_{Leib}\} = 0$ if and only if the twisted bracket

$$[x_1 + a_1, x_2 + a_2] = [x_1, x_2] + [x_1, a_2] + [a_1, x_2] + H(x_1, x_2)$$

is a Leibniz bracket, where $H := \frac{1}{3}H_{ijk}[q^i, q^j, q^k]_*$.

Definition 4.8. *Let μ be the structure defined above. We put $b_\mu := \{\mu, -\}$. This becomes a coboundary operator on $F\mathfrak{g}$. The pair $(F\mathfrak{g}, b_\mu)$ is an anticyclic cohomology complex over \mathfrak{g} .*

Finally we study a metric tensor on \mathfrak{g} . An invariant bilinear form in the category of Lie algebras is a symmetric tensor $g(.,.)$ satisfying the well-known condition,

$$g(x_1, [x_2, x_3]) = g([x_1, x_2], x_3),$$

where $[.,.]$ is an ordinary Lie bracket.

Definition 4.9. *Let $g(.,.)$ be a symmetric bilinear form on \mathfrak{g} . We call g a generalized symmetric invariant bilinear form, if*

$$g([x_1, x_2], x_3) + g(x_2, [x_1, x_3]) = g(x_1, x_2 \circ x_3), \quad (24)$$

where $x_2 \circ x_3 := [x_2, x_3] + [x_3, x_2]$.

If \mathfrak{g} is a Lie algebra as a commutative Leibniz algebra, then (24) is equal to the classical invariant condition above. In general, a symmetric bilinear form on \mathfrak{g} is a function over the cotangent bundle,

$$g = \frac{1}{2}g_{ij}[q^i, q^j]_*.$$

The bilinear form g is identified with a linear map $\tilde{g} : \mathfrak{g} \rightarrow \mathfrak{g}^*$ and satisfies (24) if and only if the graph of \tilde{g} is a subalgebra of the semi-direct product Leibniz algebra $\mathfrak{g} \ltimes \mathfrak{g}^*$.

Corollary 4.10. *g satisfies (24) if and only if $b_\mu g = \{\mu, g\} = 0$.*

Suppose that g is nondegenerate (i.e. pseudo-Euclidean metric). The inverse g^{-1} is also a function over $\mathcal{T}^*\Pi\mathcal{M}$,

$$g^{-1} = \frac{1}{2}g^{ij}[p_i, p_j]_*.$$

We denote by $X_{g^{-1}}$ the Hamiltonian vector field of g^{-1} . The canonical transformation of μ by the Hamiltonian flow $\exp(X_{g^{-1}})$ is computed as follows.

$$\exp(X_{g^{-1}})(\mu) = \mu + \{\mu, g^{-1}\} + \frac{1}{2}\{\{\mu, g^{-1}\}, g^{-1}\}.$$

If g satisfies (24), then $\exp(X_{g^{-1}})(\mu) = \mu + \{\mu, g^{-1}\}$, and vice versa. In that case, $\nu := \{\mu, g^{-1}\}$ is the second structure and $\mu + \nu$ defines a Drinfeld double in the Loday world.

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